

Quantized Non-Abelian Monopoles on S^3

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A possible electric-magnetic duality suggests that the confinement of non-Abelian electric charges manifests itself as a perturbative quantum effect for the dual magnetic charges. Motivated by this possibility, we study vacuum fluctuations around a non-Abelian monopole-antimonopole pair treated as point objects with charges $g = \pm n/2$ ($n = 1, 2, \dots$), and placed on the antipodes of a three sphere of radius R . We explicitly find all the fluctuation modes by linearizing and solving the Yang-Mills equations about this background field on a three sphere. We recover, generalize and extend earlier results, including those on the stability analysis of non-Abelian magnetic monopoles. We find that for $g \geq 1$ monopoles there is an unstable mode that tends to squeeze magnetic flux in the angular directions. We sum the vacuum energy contributions of the fluctuation modes for the $g = 1/2$ case and find oscillatory dependence on the cutoff scale. Subject to certain assumptions, we find that the contribution of the fluctuation modes to the quantum zero point energy behaves as $-R^{-2/3}$ and hence decays more slowly than the classical $-R^{-1}$ Coulomb potential for large R . However, this correction to the zero point energy does not agree with the linear growth expected if the monopoles are confined.

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I. INTRODUCTION

There are many facets to the investigation of magnetic monopoles. The most straight-forward view is that magnetic monopoles are classical solutions in certain gauge theory models [1, 2], and the possible realization of these solutions leads to important ramifications for particle physics and cosmology. Another facet is in the spirit of the original proposal by Skyrme to understand the proton as a classical solution [3, 4]. Then magnetic monopoles are themselves some versions of particles, and as suggested by the sine-Gordon and massive Thirring model equivalence [5, 6], magnetic monopoles are dual to “ordinary” electrically charged particles whereby, in some region of parameter space, it makes better sense to view the magnetic monopoles as the fundamental particles and vice versa. A tantalizing correspondence of this kind exists in grand unified inspired models where there is a one-one correspondence between the magnetic charges of stable monopoles and the electric charges of known particles such as quarks and leptons [7, 8]. Also, the duality between particles and magnetic monopoles has been shown in certain supersymmetric theories [9], substantiating the early conjecture in Ref. [10].

If magnetic monopoles are to be viewed as magnetic versions of electrically charged particles, perhaps it is simpler to study certain properties of electric charges in the magnetic sector. To follow this line of thought, we know that magnetic monopoles, like ordinary particles, can carry non-Abelian charges. Now since particles carrying non-Abelian charges are thought to be confined, perhaps non-Abelian magnetic monopoles are also confined, and it may be simpler to understand confinement by studying magnetic monopoles instead of electric particles. This argument motivated us to study quantum effects in the background of non-Abelian monopoles.

Confinement occurs due to non-perturbative effects in

the non-Abelian electric backgrounds of particles. The non-perturbative effects are supposedly due to condensation of magnetically charged objects. However, duality relates the electric and magnetic sectors with inversely related charges. Hence, if we work in the dual sector, the electric sources get replaced by magnetic charges (monopoles) and, if there is confinement, it would be due to the condensation of electric charges. We emphasize that the exchange of (weak, electric) and (strong, magnetic) is simply a hypothesis. However, within this hypothesis, magnetic monopoles should get confined by perturbative quantum effects in the electric sector. The quantum corrections to the magnetic field of the monopole should be calculable using a perturbative expansion in the electric coupling.

It is to be noted that the effect we are looking for is distinct from the confinement of magnetic monopoles due to spontaneous symmetry breaking by the condensation of an order parameter. We do not have extra scalar fields that take on a vacuum expectation value to break the non-Abelian symmetry and hence do not have any topological strings that confine the monopoles. Instead we are hoping to see evidence for a quantum string that confines the monopoles.

Specifically, we aim to find the one-loop quantum contribution to the energy of a non-Abelian monopole and an anti-monopole as a function of their separation (see Fig. 1). In this, we have to assume a background classical field configuration for the monopoles and this is constructed by linearly superposing the spherically symmetric classical solutions. If the quantum corrected energy grows faster than the separation, it would indicate that the choice of spherically symmetric magnetic field configurations for each monopole is not preferred since, if all the magnetic flux is confined to a tube, the energy only grows linearly with separation.

A monopole and an antimonopole in R^3 is a compli-

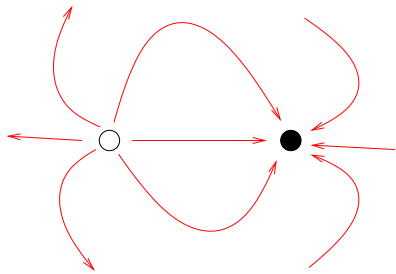


FIG. 1: A classical monopole and an antimonopole in R^3 and their magnetic field lines. The system has little symmetry, making it difficult to evaluate quantum fluctuations in this background.

cated background in which to study fluctuation modes which then have to be summed up to find the quantum corrected energy (see though [11]). Hence we have chosen to study a monopole and an antimonopole placed at the antipodal points of an S^3 , as shown in Fig. 2. The classical magnetic field of the monopoles is then spherically symmetric, and mitigates the difficulty of finding the fluctuation modes. For the time being, we will simply focus on the analysis on S^3 , and hope that a mapping to R^3 can be made at a later stage, or perhaps a similar analysis will be found to be feasible directly in R^3 .

Our first task is to construct non-Abelian monopoles. The simplest model with such monopoles involves the symmetry breaking

$$SU(3) \rightarrow [SU(2) \times U(1)]/Z_2$$

and the fundamental monopoles have non-trivial $SU(2)$ and $U(1)$ charge. The symmetry breaking is achieved by giving a vacuum expectation value to a field Φ transforming in the adjoint representation of $SU(3)$. The monopoles have a regular core and are known quite explicitly since they are basically $SU(2)$ 't Hooft-Polyakov monopoles embedded within $SU(3)$ [12].

Analysis of fluctuations around monopoles arising from the $SU(3)$ model is very complicated since it involves both scalar and gauge fields. Also, the structure of the core will be important in determining the fluctuation modes. So we have chosen to study non-Abelian *Dirac* monopoles. In this case, described more explicitly in Sec. II, the monopole core is singular but the external magnetic field for $g = 1/2$ is identical to the $SU(2)$ part of the fundamental $SU(3)$ magnetic monopole. Ignoring the core structure does mean that we need to impose boundary conditions at the location of the monopole by hand, an issue that we discuss in Appendix B.

In solving for the angular part of the fluctuation modes we are aided by the early work of Wu and Yang [13] who found monopole scalar harmonics, Brandt and Neri [14], who studied the stability of non-Abelian monopoles, Olsen et al [15] and Weinberg [16], who constructed monopole vector harmonics (also [14]). The radial problem in R^3 has been set up in Ref. [15] though with the

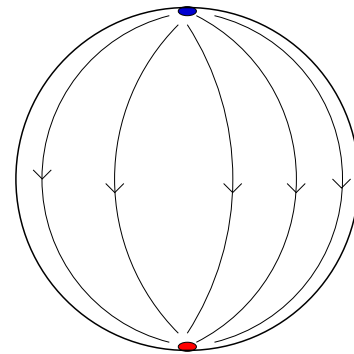


FIG. 2: A monopole and an antimonopole on S^3 and the classical, spherically symmetric field lines. (One dimension of the S^3 has been suppressed.)

limited goal of finding a bound state solution. Instead our analysis requires us to evaluate *all* solutions of the angular and radial problems on S^3 , in the background of an monopole-antimonopole pair (and not just a single monopole).

To evaluate the one-loop quantum correction to the energy of minimal charge monopoles with $g = 1/2$ on S^3 , we follow the procedures developed in [17] (see [18, 19] for more detailed expositions). We find all the eigenfrequencies of the fluctuation modes and then sum them up to an ultraviolet cutoff. We do not face some of the subtleties in the quantization of solitons since we are in a compact space and so all the fluctuation modes are discrete. However, we treat the monopoles as point-like (*i.e.* non-Abelian Dirac monopoles) and have no control on the renormalization of its mass for example. But we are primarily interested in the dependence of the quantum corrections to the energy as a function of the monopole-antimonopole separation, and assuming that the mass of the Dirac monopole has no dependence on the size of the S^3 , this is given by the dependence of the sum of zero point fluctuations on R , which we can indeed calculate.

We begin by describing the background of non-Abelian monopoles on S^3 in Sec. II. This is followed in Sec. III by a derivation of the linearized fluctuation equations. Here we see that the non-Abelian gauge fluctuations split up into 4 types of excitations depending on whether they are uncharged or charged and whether they are scalar or vector in nature. The uncharged scalar and vector sectors do not feel the non-Abelian monopole background but do play a role in the zero point energy, and we construct all of them on S^3 . Charged scalar fluctuation modes in the background of a single monopole in R^3 have been considered by Wu and Yang [13] leading to the Wu-Yang spherical harmonics. Here we also solve for the radial dependence and find the full set of charged scalar harmonics on S^3 in the background of the monopole-antimonopole. Similarly vector spherical harmonics in the background of a monopole in R^3 have been constructed in Refs. [14, 15, 16]. Here we find all the charged vector spherical harmonics on the monopole-

antimonopole background in S^3 , including the radial dependence of the eigenfunctions. In Sec. IV we have summarized the technical results for the derivation of the eigenvalues and eigenfunctions that are shown in detail in the Appendices. For the $g = 1/2$ monopole, the sum over vacuum fluctuations is done in Sec. VII. We close the paper with discussions in Sec. VIII.

The evaluation of the fluctuation modes is quite technical and so it is helpful to summarize our findings qualitatively at this stage for the reader who may not be interested in details. As far as the stability and possible confinement of non-Abelian monopoles is concerned, the charged vector modes are of particular interest. These vector modes interact with the monopole background via their charge and also by the Zeeman coupling of their spin to the background magnetic field. In addition, they experience the curvature of the S^3 . If $g \geq 1$, where $g = eq/4\pi$ and q is the magnetic monopole charge, we find that there is an unstable mode in the spectrum of fluctuations, implying that the spherically symmetric background is unstable. This is consistent with the result in [14] but we also find the unstable mode explicitly and show that it tends to squeeze the monopole flux in the angular directions. For $g = 1/2$, however, this classically unstable mode is absent. In this case, we find the eigenvalues and eigenfunctions for all the fluctuation modes. We then evaluate the contribution of these quantum zero point fluctuations to the energy of the monopole background, as compared to the zero point energy of the trivial background, by summing up the energies of all the individual modes. If we sum up to Γ_c modes, we find that this contribution to the vacuum energy oscillates as a function of Γ_c . Averaged over the oscillations the contribution is proportional to $-\Gamma_c^{1/3}/R$. If we impose an ultraviolet cutoff on the momentum, this corresponds to a $-R^{-2/3}$ dependence of the vacuum energy on the radius of the S^3 which is directly related to the separation of the monopole-antimonopole pair.

Additional contributions to the vacuum energy will arise from renormalization of the parameters, in this case the mass of the monopoles. We have taken our monopoles to be point-like and hence do not have control over these extra contributions. In any case, the $-R^{-2/3}$ dependence is to be contrasted with a $+R$ growth in the energy if the monopole and antimonopole are confined by a flux tube. Hence the energy of the spherically symmetric contribution does not become larger than the energy of a flux tube for some critical R and the spherically symmetric configuration is of lower energy. So we conclude that the $g = 1/2$ monopoles on S^3 are not confined, at least for reasons of vacuum energy. However, the quantum zero point energy may still turn out to be larger than the classical $-1/R$ Coulomb energy for monopole-antimonopole separation larger than some critical value.

II. MONOPOLE BACKGROUND

We consider a Yang-Mills field on a three sphere S^3 of radius R . The S^3 metric in polar coordinates (χ , θ , and ϕ) is

$$ds^2 = -dt^2 + R^2 \left(d\chi^2 + \sin^2 \chi \left[d\theta^2 + \sin^2 \theta d\phi^2 \right] \right) \quad (1)$$

The Yang-Mills equations of motion are

$$\nabla_\mu F^\mu{}_\alpha + ie [A_\mu, F^\mu{}_\alpha] = 0, \quad (2)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ie [A_\mu, A_\nu]$. The curvature of space is taken into account via the covariant derivative ∇_μ which acts on vector fields as $\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma_{\mu\sigma}^\nu A^\sigma$. For simplicity we will consider only $SU(2)$ Yang-Mills fields in this paper. The generalization to $SU(N)$ is straightforward.

An exact solution to the Yang-Mills equations (2) is

$$A_\phi = q \left(\frac{1 - \cos \theta}{4\pi} \right) \hat{\sigma}^{(3)}, \quad 0 \leq \theta < \pi \quad (3)$$

where $\hat{\sigma}^{(3)}$ is the direction in group space, and q is the magnetic charge. Note that the gauge potential is not defined at $\theta = \pi$. As discussed in [13], we also need to define the gauge potential on a second coordinate patch

$$A_\phi = -q \left(\frac{1 + \cos \theta}{4\pi} \right) \hat{\sigma}^{(3)}, \quad 0 < \theta \leq \pi \quad (4)$$

The gauge fields in the region of overlap of the two coordinate patches are related by a gauge transformation that is well-defined provided

$$g = \frac{eq}{4\pi} = \frac{n}{2}, \quad n = 0, \pm 1, \pm 2, \dots \quad (5)$$

The background field strength can be calculated from the gauge potential and is

$$F_{\theta\phi} = q \frac{\sin \theta}{4\pi} \hat{\sigma}^{(3)}. \quad (6)$$

and corresponds to the canonical monopole magnetic field

$$(\vec{B})_\chi = \frac{1}{r^2 \sin \theta} F_{\theta\phi} = \frac{q}{4\pi} \frac{1}{r^2} \hat{\sigma}^{(3)}. \quad (7)$$

where $r \equiv \sin \chi$.

This exact solution corresponds to a monopole at the north pole ($\chi = 0$) of the S^3 and an antimonopole at the south pole ($\chi = \pi$). The solution is singular at the location of the poles, as is the case for Dirac monopoles. This issue could be eliminated by considering regular 't Hooft-Polyakov monopoles [1, 2], but it would complicate the analysis significantly and would not change the long distance behavior of the non-Abelian system which is our primary interest here.

III. LINEARIZED YANG MILLS THEORY

We now consider a linear perturbation a_μ around the background monopole solution A_μ . Perturbing Eq. (2) and keeping terms that are first order in a gives

$$\begin{aligned} g^{\mu\nu} \Big\{ & (\partial_\mu \partial_\nu a_\alpha - \Gamma_{\mu\nu}^\lambda \partial_\lambda a_\alpha) + \Gamma_{\mu\alpha}^\lambda (\partial_\lambda a_\nu - \partial_\nu a_\lambda) \\ & + ie \left([\partial_\mu a_\nu, A_\alpha] + [\partial_\mu A_\nu, a_\alpha] + 2[A_\mu, \partial_\nu a_\alpha] \right. \\ & - \Gamma_{\mu\nu}^\lambda \left([a_\lambda, A_\alpha] + [A_\lambda, a_\alpha] \right) \\ & - \Gamma_{\mu\alpha}^\lambda \left([a_\nu, A_\lambda] + [A_\nu, a_\lambda] \right) + 2[a_\mu, F_{\nu\alpha}] \Big) \\ & \left. + (ie)^2 \left([A_\mu, [A_\nu, a_\alpha]] - [A_\alpha, [A_\mu, a_\nu]] \right) \right\} \\ & + (\partial_\alpha g^{\mu\nu}) \left(\partial_\mu a_\nu - \Gamma_{\mu\nu}^\lambda a_\lambda + ie[A_\mu, a_\nu] \right) \\ & - g^{\mu\nu} (\partial_\alpha \Gamma_{\mu\nu}^\lambda) a_\lambda = 0. \end{aligned} \quad (8)$$

We work in the fixed background gauge [21],

$$g^{\mu\nu} (\partial_\mu a_\nu - \Gamma_{\mu\nu}^\lambda a_\lambda + ie[A_\mu, a_\nu]) = 0. \quad (9)$$

The equations of motion can be cast into a more friendly form by decomposing the perturbation in terms of its group components, as well as separating the time and spatial parts of the 4-vector,

$$\begin{aligned} a_\mu &= \sum_{k=\pm,3} a_\mu^{(k)} \hat{\sigma}^{(k)} \\ a_\mu^{(k)} &= \left(a_t^{(k)}, \mathbf{a}^{(k)} \right). \end{aligned} \quad (10)$$

Here $k = \pm, 3$ is the group index, $\hat{\sigma}^{(k)}$ are the Pauli matrices with $\hat{\sigma}^{(\pm)} \equiv \hat{\sigma}^{(1)} \pm i\hat{\sigma}^{(2)}$. With this decomposition, the equations decouple into 4 sectors: the uncharged time component which behaves as a scalar $a_t^{(3)}$, the scalar charged component $a_t^{(+)}$, the uncharged spatial vector $\mathbf{a}^{(3)}$, and the charged vector $\mathbf{a}^{(+)}$. The equations of motion can then be schematically written the following way:

$$\left(-R^2 \partial_t^2 + \Delta^{(3)} \right) a_t^{(3)} = 0 \quad (11)$$

$$\left(-R^2 \partial_t^2 + \Delta^{(3)} \right) \mathbf{a}^{(3)} = 0 \quad (12)$$

$$\left(-R^2 \partial_t^2 + \Delta^{(+)} \right) a_t^{(+)} = 0 \quad (13)$$

$$\left(-R^2 \partial_t^2 + \Delta^{(+)} \right) \mathbf{a}^{(+)} = 0. \quad (14)$$

Here Δ and Δ are the scalar and vector Laplacians respectively, and in the $(+)$ direction the derivatives are gauged, $\partial_i \rightarrow \partial_i - ieA_i$. The vector Laplacians take into account the coupling to the curvature of the S^3 sphere, via the Ricci scalar, and in the $(+)$ direction there is also a Zeeman-like coupling between the background monopole and the perturbations. We have not

written the equations for the $(-)$ scalar and vector components as they are simply the complex conjugates of the Eqs. (13) and (14) of the $(+)$ direction.

The gauge condition (9) decouples into the neutral and the charged sectors, but it does mix between the time component and the spatial vector. It can be schematically written as

$$-R \partial_t a_t^{(3)} + \nabla^{(3)} \mathbf{a}^{(3)} = 0 \quad (15)$$

$$-R \partial_t a_t^{(+)} + \nabla^{(+)} \mathbf{a}^{(+)} = 0. \quad (16)$$

The explicit equations can be found in Appendix A. The boundary conditions to which they are subject are discussed in Appendix B.

IV. ENUMERATION OF EIGENMODES

Constructing the solutions to the linearized Yang-Mills equations is a long yet subtle exercise in classical mathematical analysis. The details are given in Appendix B; here we simply enumerate the solutions and their associated frequencies. At first we will simply list all the solutions to the equations for the fluctuations. In Sec. IV G we will impose the gauge conditions and also eliminate modes that are pure gauge. Finally we will be left with the physical modes and these are summarized in Sec. V.

A. Neutral scalar sector

In this sector we compute the eigenmodes of the scalar Laplacian on S^3 . The modes are derived in Appendix B by separation of variables and again in Appendix C by the use of group theory; the latter derivation exploits the four-dimensional rotational symmetry of the problem and shows how the solutions organize into multiplets that are irreducible representations of the $so(4)$ algebra. The resulting solutions are:

$$a_t^{(3)} = Y_{JM}(\theta, \phi) \xi_{n-J, J}(\chi) e^{i\omega t}. \quad (17)$$

Here

$$\xi_{n-J, J}(\chi) \equiv \sin^J(\chi) G_{n-J}^{J+\frac{1}{2}}(\cos \chi) \quad (18)$$

where $Y_{JM}(\theta, \phi)$ are the familiar spherical harmonics and G are the Gegenbauer polynomials [20]. The quantum numbers span the range $n = 1, 2, 3, \dots$ and $J = 0, 1, 2, \dots, n$ and $M = -J, \dots, +J$. The energy of this solution is given by

$$\Lambda \equiv \omega R = \sqrt{n(n+2)} \quad (19)$$

B. Charged scalar sector

In this sector we must compute the eigenmodes of the scalar Laplacian on S^3 that has been minimally coupled

to the gauge field of the monopole antimonopole pair. The solutions are obtained by separation of variables:

$$a_t^{(+)} = W_{JM}(\theta, \phi) \xi_{n-J, \alpha}(\chi) e^{i\omega t} \quad (20)$$

where

$$\alpha = -\frac{1}{2} + \sqrt{\left(J + \frac{1}{2}\right)^2 - g^2}. \quad (21)$$

$W_{JM}(\theta, \phi)$ are the Wu-Yang monopole spherical harmonics [13] and $\xi_{n-J, \alpha}$ is given by

$$\xi_{n-J, \alpha} \equiv \sin^\alpha(\chi) G_{n-J}^{\alpha+\frac{1}{2}}(\cos \chi). \quad (22)$$

The quantum numbers satisfy $n = g, g+1, g+2, \dots$ and $J = g, g+1, g+2, \dots, n$ and $M = -J \dots J$. The energy of this solution is given by

$$\Lambda = \sqrt{(n-J+\alpha)(n-J+\alpha+2)} \quad (23)$$

Note that $\alpha = J$ for $g = 0$ and the charged solutions reduce to the neutral ones as expected.

C. Neutral vector sector

In essence we are looking for the eigenmodes of the vector wave equation on S^3 here. Since this is a vector equation we expect three sets of solutions corresponding to different states of polarization. One set of solutions, Eq. (24), may be constructed simply by taking the gradient of the neutral scalar eigenmodes, Eq. (17). The third set, Eq. (26), is obtained by taking the curl of the second. The second set of solutions, Eq. (25) below, are obtained by looking for transverse solutions that have no radial χ component, as discussed in Appendix B.

First solution:

$$\begin{aligned} a_\chi^{(3)}(1) &= Y_{JM}(\theta, \phi) \partial_\chi \xi_{n-J, J}(\chi) e^{i\omega t} \\ a_\theta^{(3)}(1) &= \partial_\theta Y_{JM}(\theta, \phi) \xi_{n-J, J}(\chi) e^{i\omega t} \\ a_\phi^{(3)}(1) &= \partial_\phi Y_{JM}(\theta, \phi) \xi_{n-J, J}(\chi) e^{i\omega t} \\ \Lambda &= \sqrt{n(n+2)} \end{aligned} \quad (24)$$

Second solution:

$$\begin{aligned} a_\chi^{(3)}(2) &= 0 \\ a_\theta^{(3)}(2) &= \frac{1}{\sin \theta} [\partial_\phi Y_{JM}(\theta, \phi)] \sin \chi \xi_{n-J, J}(\chi) e^{i\omega t} \\ a_\phi^{(3)}(2) &= -\sin \theta [\partial_\theta Y_{JM}(\theta, \phi)] \sin \chi \xi_{n-J, J}(\chi) e^{i\omega t} \\ \Lambda &= n+1 \end{aligned} \quad (25)$$

Third solution:

$$\begin{aligned} a_\chi^{(3)}(3) &= J(J+1) Y_{JM}(\theta, \phi) \frac{1}{\sin \chi} \xi_{n-J, J}(\chi) e^{i\omega t} \\ a_\theta^{(3)}(3) &= \partial_\theta Y_{JM}(\theta, \phi) \partial_\chi [\sin \chi \xi_{n-J, J}(\chi)] e^{i\omega t} \\ a_\phi^{(3)}(3) &= \partial_\phi Y_{JM}(\theta, \phi) \partial_\chi [\sin \chi \xi_{n-J, J}(\chi)] e^{i\omega t} \\ \Lambda &= n+1 \end{aligned} \quad (26)$$

The quantum numbers span the range $n = 1, 2, 3, \dots$, $J = 1, 2, \dots, n$ and $M = -J, \dots, J$ for the second and third solution. For the first solution the allowed range is $n = 1, 2, 3, \dots$, $J = 0, 1, 2, \dots, n$ and $M = -J, \dots, J$; in other words the value $J = 0$ is also allowed.

D. Charged vector sector

There is a close parallel between the solutions in the neutral and charged vector sectors. With one exception, the charged solutions can be obtained from the neutral solutions by the replacement of spherical harmonics Y_{JM} with Wu-Yang harmonics W_{JM} , of $J(J+1)$ with $J(J+1) - g^2$, and the minimal coupling substitution $\partial_\phi \rightarrow \partial_\phi - iA_\phi \equiv D_\phi$ where A_ϕ is the background vector potential. The exception is that the second solution involves an additional term proportional to g as explained in Appendix B.

First Solution:

$$\begin{aligned} a_\chi^{(+)}(1) &= W_{JM}(\theta, \phi) \partial_\chi \xi_{n-J, \alpha}(\chi) e^{i\omega t} \\ a_\theta^{(+)}(1) &= \partial_\theta W_{JM}(\theta, \phi) \xi_{n-J, \alpha}(\chi) e^{i\omega t} \\ a_\phi^{(+)}(1) &= D_\phi W_{JM}(\theta, \phi) \xi_{n-J, \alpha}(\chi) e^{i\omega t} \\ \Lambda &= \sqrt{(n-J+\alpha)(n-J+\alpha+2)} \end{aligned} \quad (27)$$

Second Solution:

$$\begin{aligned} a_\chi^{(+)}(2) &= 0 \\ a_\theta^{(+)}(2) &= \sin \chi \xi \left([J(J+1) - g^2] \frac{i}{\sin \theta} D_\phi W + g i \partial_\theta W \right) \\ a_\phi^{(+)}(2) &= \sin \chi \xi (-[J(J+1) - g^2] i \sin \theta \partial_\theta W + g D_\phi W) \\ \Lambda &= n+1 - J + \alpha \end{aligned} \quad (28)$$

For brevity we have written $\xi_{n-J, \alpha}(\chi)$ as ξ and $W_{JM}(\theta, \phi)$ as W .

Third solution:

$$\begin{aligned} a_\chi^{(+)}(3) &= [J(J+1) - g^2] W_{JM} \frac{1}{\sin \chi} \xi_{n-J, \alpha} e^{i\omega t} \\ a_\theta^{(+)}(3) &= \partial_\theta W_{JM}(\theta, \phi) \partial_\chi [\sin \chi \xi_{n-J, \alpha}(\chi)] e^{i\omega t} \\ a_\phi^{(+)}(3) &= D_\phi W_{JM}(\theta, \phi) \partial_\chi [\sin \chi \xi_{n-J, \alpha}(\chi)] e^{i\omega t} \\ \Lambda &= n+1 - J + \alpha \end{aligned} \quad (29)$$

In the first and third families the quantum numbers span the range $n = g, g+1, \dots$, $J = g, g+1, \dots, n$ and $M = -J, \dots, J$. For the second family $n = g+1, g+2, \dots$, $J = g+1, \dots, n$ and $M = -J, \dots, J$.

The solutions given here are valid in both the north pole gauge, Eq. (3), and the south pole gauge, Eq. (4), if we interpret A_ϕ and W_{JM} appropriately.

E. Exceptional modes

In the charged vector sector there is a new set of modes provided $g \geq 1$ and these have no counterpart in the neu-

tral sector or for $g = 1/2$. These exceptional modes occur for $J = g - 1$ and $M = -J, \dots, +J$. The radial quantum number n becomes continuous as does the frequency spectrum. The construction of these modes is described in Appendix B where we also describe the radial functions $\xi(\chi)$. The explicit form of the modes is:

$$\begin{aligned} a_\chi &= 0 \\ a_\theta &= \xi \sin \chi \exp[i(M+g)\phi] (\sin \theta)^{g+M-1} (1 + \cos \theta)^{-M} \\ a_\phi &= i\xi \sin \chi \exp[i(M+g)\phi] (\sin \theta)^{g+M} (1 + \cos \theta)^{-M}. \end{aligned} \quad (30)$$

The frequencies of these solutions form a continuum, $\Lambda^2 \in (-\infty, +\infty)$.

It may seem surprising that the spectrum is a continuum on a finite space S^3 . This result is an artifact of treating the monopoles as point objects. As explained in Appendix B this leads to a singular potential in the mode equation for the $J = g - 1$ angular momentum channel; the singular potential in turn is responsible for the continuum character of the spectrum. If we worked with a model in which the monopoles had structure the singularity would be softened and we would presumably obtain a discrete spectrum. However we still expect that there would be both bound and unbound modes with Λ^2 less than and greater than zero respectively. The precise spectrum would depend on the assumed structure of the monopole core. Fortunately we do not need to determine the exact spectrum for the problems we wish to consider in this paper.

F. Zero modes

So far we have been looking at modes which are completely regular at the poles $\chi = 0, \pi$ of the S^3 . If we admit modes that diverge at the poles, we can construct zero modes that can be interpreted as translations of the monopoles.

If we rotate the monopole and antimonopole by a small amount while keeping them antipodal the change in the field configuration will be a zero mode—a zero frequency solution to the linearized Yang-Mills equations. It is a solution because the original and displaced monopole fields are both exact solutions to the full Yang-Mills equations; therefore their difference should be a solution to the linearized equation for small displacement. The frequency vanishes because both original and displaced configurations have the same energy. These zero mode solutions are constructed explicitly in Appendix B. There are three independent zero mode solutions corresponding to the three orthogonal directions in which we can infinitesimally displace the monopole-antimonopole pair. Explicitly these solutions are given by:

First Zero Mode:

$$\begin{aligned} a_\chi^{(3)} &= 0 \\ a_\theta^{(3)} &= 0 \\ a_\phi^{(3)} &= \frac{\cos \chi}{\sin \chi} (1 - \cos^2 \theta). \end{aligned} \quad (31)$$

Second Zero Mode:

$$\begin{aligned} a_\chi^{(3)} &= \frac{1}{\sin^2 \chi} \left(\frac{1 - \cos \theta}{\sin \theta} \right) \sin \phi \\ a_\theta^{(3)} &= \frac{\cos \chi}{\sin \chi} \left(\frac{\cos \theta - \cos^2 \theta}{\sin^2 \theta} \right) \sin \phi \\ a_\phi^{(3)} &= -\frac{\cos \chi}{\sin \chi} \left(\frac{1 - \cos \theta}{\sin \theta} - \sin \theta \cos \theta \right) \cos \phi \end{aligned} \quad (32)$$

Third Zero Mode:

$$\begin{aligned} a_\chi^{(3)} &= \frac{1}{\sin^2 \chi} \left(\frac{1 - \cos \theta}{\sin \theta} \right) \cos \phi \\ a_\theta^{(3)} &= \frac{\cos \chi}{\sin \chi} \left(\frac{\cos \theta - \cos^2 \theta}{\sin^2 \theta} \right) \cos \phi \\ a_\phi^{(3)} &= \frac{\cos \chi}{\sin \chi} \left(\frac{1 - \cos \theta}{\sin \theta} - \sin \theta \cos \theta \right) \sin \phi \end{aligned} \quad (33)$$

The zero modes lie in the uncharged sector because they correspond to the difference of two field configurations with the same charges. Note that the zero modes are singular along the line $\theta = \pi$. This is because we have worked in the north pole gauge Eq. (3). If we had worked in the south pole gauge, Eq. (4) the solutions would have been singular along $\theta = 0$. The zero modes did not arise in the previous analysis of the neutral sector because we had previously restricted attention to solutions that were globally regular.

As the three zero modes listed above are associated with displacement of the monopole and the antimonopole, they are not present when the background is trivial. It is worth noting that there is another solution with zero frequency, not associated with displacement, and therefore present also for the trivial case: this is the neutral scalar sector solution (19) with $n = J = M = 0$.

G. Gauge condition

The gauge conditions, Eqs. (15) and (16), do not mix the neutral and the charged sectors but, unlike the equations of motion, do mix the scalar and vector components. Using the solutions of the previous section, we now find linear combinations of those solutions that satisfy the gauge conditions. In other words, we find 4-vectors of the form $(\alpha a_t^{(k)}, \beta_j \mathbf{a}^{(\mathbf{k})}(j))$, where α and β_j are constants, and $j = 1, 2, 3$ labels the 3 vector solutions we have found. The coefficients α and β_j are determined by inserting these linear combinations into (15) or (16).

We find three solutions, $a_\mu^{(k)} = (i\omega a_t^{(k)}, \mathbf{a}^{(k)}(1)), (0, \mathbf{a}^{(k)}(2)),$ and $(0, \mathbf{a}^{(k)}(3))$ [24]. However, the first combination, $(i\omega a_t^{(k)}, \mathbf{a}^{(k)}(1))$ is pure gauge. This can be easily seen as it is the gradient of a scalar function. As expected, there is no longitudinal mode for a massless gauge field. Discarding this solution, we are finally left with 2 physical (transverse) modes for each sector.

A similar analysis of the zero modes shows that the three solutions, Eqs. (31)-(33) combined with $a_t^{(3)} = 0$, satisfy the gauge condition and are physical. The last zero mode, (19) with $n = J = M = 0$, is pure gauge for the monopole-pair background, as well as for the trivial background.

V. SUMMARY OF PHYSICAL MODES

After eliminating the combinations of eigenmodes that do not satisfy the gauge condition and also the combinations that are pure gauge, we are left with the following physical eigenmodes and energies.

Trivial Background: In the absence of the monopole-antimonopole pair all three directions in group space (\pm and 3) look alike, and each of them has 2 physical eigenmodes given by Eqs. (25) and (26) with $a_t^k = 0$ (where $k = \pm, 3$). Both eigenmodes have the same frequency,

$$\Lambda = n + 1, \quad (34)$$

where $n = 1, 2, 3, \dots, J = 1, 2, \dots, n$ and $M = -J, \dots, J$. There aren't any physical zero modes.

Monopole-antimonopole background: The neutral sector (which is direction (3) in group space) has 3 zero modes with $\Lambda = 0$, and two physical eigenmodes, Eqs. (25) and (26), both with energy

$$\Lambda = n + 1, \quad (35)$$

where $n = 1, 2, 3, \dots, J = 1, 2, \dots, n$ and $M = -J, \dots, J$. The (+) and (-) directions of group space are similar, and each has 2 physical eigenmodes given by Eqs. (28) and (29), again with $a_t^{(\pm)} = 0$. The energy eigenvalue for both modes is

$$\Lambda = n + 1 - J + \alpha \quad (36)$$

where $n = g + 1, \dots, J = g + 1, \dots, n$ and $M = -J, \dots, +J$ for the physical solution corresponding to Eq. (28), and $n = g, g + 1, \dots, J = g, g + 1, \dots, n$ and $M = -J, \dots, +J$ for the physical solution corresponding to Eq. (29). The parameter α is defined in Eq. (21). For $g \geq 1$ there is an additional exceptional mode, given by Eq. (30). The eigenfrequencies of these solutions form a continuum, $\Lambda^2 \in (-\infty, +\infty)$.

VI. CLASSICAL STABILITY

A key consideration is whether the modes are stable. The diagnostic for stability is whether the frequency of a mode is real or imaginary, or equivalently, whether the frequency squared is positive or negative.

We find that the lowest frequency for $g = 1/2$, which occurs in the charged vector sector for $n = 1/2$ and $J = 1/2$, is real and positive:

$$\min(\Lambda) = \frac{\sqrt{3} + 1}{2} = 1.37 \quad (37)$$

(see Eq. (28) or (29)). Hence a monopole with $g = 1/2$ is classically stable.

In the case of $g \geq 1$ monopoles there is a continuum of exceptional modes, Eq. (30), that have $\Lambda^2 < 0$ and are hence unstable. This shows that monopoles with $g \geq 1$ are classically unstable, a result first obtained by Brandt and Neri [14].

To visualize the instability of $g \geq 1$ monopoles, consider the $g = 1$ case. Then we have $J = g - 1 = 0$ and hence $M = 0$, and we find

$$f_{\theta\phi} \equiv \partial_\theta a_\phi - \partial_\phi a_\theta = -i\xi \sin \chi e^{i\phi} (1 - \cos \theta) \quad (38)$$

Therefore the unstable mode develops a radial magnetic field component at $\theta = \pi$ but none at $\theta = 0$, suggesting that the instability of $g = 1$ monopoles is toward a spherically asymmetric configuration.

VII. QUANTUM EFFECTS

We now turn to the quantum correction to the energy of a monopole antimonopole pair. We consider only the case $g = 1/2$ since for higher g values the configuration is classically unstable. Our objective is to analyze the divergent zero-point energy in the monopole background. From this divergent contribution we make inferences about how the energy of the configuration scales with the distance R between the monopole-antimonopole pair. A faster than linear growth would suggest confinement. On the other hand if the energy scales more slowly or if it decreases, but slower than $1/R$, it is still significant because it suggests that at large distance the energy of a pair is dominated by quantum effects.

We aim to compute E_Q , the difference in the zero-point energy of the monopole-antimonopole and the zero-point energy in the absence of the monopole-antimonopole,

$$E_Q R = \sum_{\text{modes}} \Lambda \Big|_{\text{m}\bar{\text{m}}} - \sum_{\text{modes}} \Lambda \Big|_{\text{trivial}} \quad (39)$$

where the $\text{m}\bar{\text{m}}$ subscript refers to Λ in the background of the monopole-antimonopole pair, and the trivial subscript refers to Λ when there are no monopoles *i.e.* the trivial vacuum. Once we have E_Q , we wish to study its dependence on the number of summed modes Γ_c , which

we also relate to the distance between the monopoles by $\Gamma_c = PR$ where P is an ultraviolet momentum cut-off.

Ideally we would sum over modes in the $m\bar{m}$ sector that map onto the modes in the trivial sector. Unfortunately it is not clear there is such a mapping. We might consider bringing the monopole and antimonopole together until they annihilate and watch the modes evolve through this process. Unfortunately the modes are highly degenerate when the monopole and antimonopole are antipodal or coincident and so it is impossible to establish a continuity between modes at these two extreme points. Alternatively one could imagine keeping the monopole and antimonopole fixed and study how the modes evolve as g is turned off. The problem is that for consistency g must be half-integer quantized so we cannot continuously vary g . Although physically variation of g is impossible, we cannot rule out that there is some way to establish a mapping by analytically continuing g from $1/2$ to zero.

For the present work, we have subtracted the mode energies by arranging the modes (indexed with k) in ascending order in energy, $\Lambda(k+1) \geq \Lambda(k)$, and fixing the number of summed modes Γ_c ,

$$E_Q R = \sum_{k=1}^{\Gamma_c} \Lambda(k) \Big|_{m\bar{m}} - \sum_{k=1}^{\Gamma_c} \Lambda(k) \Big|_{\text{trivial}} \quad (40)$$

The ordering of the modes according to the energy is as follows. In the trivial background the mode energy only depends on the n quantum number and we simply order the modes according to n . In the $m\bar{m}$ background, the energy of the mode is $n+1-J+\alpha$. It is easy to see that the combination $-J+\alpha$ lies in the interval $[-1/2, 0)$ and is monotonically increasing with increasing J . This implies that the J dependence of the mode energies simply gives a small splitting among different states having the same n quantum number. Therefore it is sufficient to order the modes primarily according to increasing n and then, for the same values of n , according to increasing J .

It is obvious that the charged sector modes are different depending on whether they are computed about the trivial background or the background of the monopole antimonopole pair. Thus these modes will contribute to the E_Q which is the difference in the zero point energy with the monopole background and the zero point energy with a trivial background. A more subtle point is that the neutral sector modes are also different about the monopole background. In this case there are three zero modes that are absent in the fluctuation spectrum about the trivial background, so the first mode of the neutral sector in the different backgrounds contributes differently,

$$\Lambda^{(3)}(1) \Big|_{m\bar{m}} = 0, \quad \Lambda^{(3)}(1) \Big|_{\text{trivial}} = 1. \quad (41)$$

Thus the neutral modes also contribute to the difference in zero point energies E_Q that we wish to calculate.

It is possible to obtain analytical bounds on E_Q and these are derived in Appendix D. Here we show our numerical results for E_Q , obtained after summing Eq. (40).

Neutral sector		Charged sector	
n	multiplet	n	multiplet
1	6	1/2	4
2	16	3/2	12
3	30	5/2	24
k	$2k^2 + 4k$	$k + 1/2$	$2k^2 + 6k + 4$

TABLE I: The first few n quantum numbers of the neutral (left) and charged (right) sectors, and the number of modes associated with that quantum number.

Fig. 3 shows a plot of $E_Q R$ versus Γ_c , plotted in solid line. The growing oscillatory behavior in Fig. 3 makes it hard to interpret the result for E_Q . The oscillations originate due to the difference in the degeneracy structure of the $m\bar{m}$ and of the trivial case. The total angular momentum of the neutral sector is integer, $J = 1, 2, \dots, n$. For the charged sector, the total angular momentum is half integer, $J = 1/2, 3/2, \dots, n$. Ignoring the fine splitting of J values in the charged sector, the n th multiplet has $\sum_J 2(2J+1)$ modes. While the form is similar for both the neutral and charged sector, the relevant J span is different, resulting in multiplets of different sizes. The first few n 's and their associated multiplets of both sectors are listed in Table I. The result is that each sector reaches saturation of an energy level at a different value of Γ_c . The energy sum is dominated by the charged sector for the monopole-antimonopole background and by the neutral sector for the trivial background, creating the zig-zag effect of Fig. 3. The enveloping cone, plotted with dotted lines, is calculated at Γ_c values where either the neutral or the charged sectors are saturated and jumping an energy level. Fig. 3 also shows the average of E_Q (dashed line). As each oscillation is almost linear, we average only the extremal points, which are on the enveloping cone. Naming the extremal points (Γ_i, E_{Qi}) , the average is calculated according to

$$\Gamma_i^{(\text{average})} = \frac{1}{4} (\Gamma_{i-1} + 2\Gamma_i + \Gamma_{i+1}) \quad (42)$$

$$E_{Qi}^{(\text{average})} = \frac{1}{4} (E_{Qi-1} + 2E_{Qi} + E_{Qi+1}) \quad (43)$$

In Fig. 4 we plot average of $E_Q R$ on a log-log plot and find $\langle E_Q R \rangle \propto -\Gamma_c^{1/3}$. We have also plotted the enveloping curves themselves on a log-log plot in Fig. 5. The result is that the amplitude of the oscillations grow in proportion to $\Gamma_c^{2/3}$; the lower enveloping curve $\propto -\Gamma_c^{2/3}$, and the upper bound $\propto +\Gamma_c^{2/3}$.

A more complete renormalization analysis would require us to evaluate other divergent contributions to the energy of the configuration and to extract from them a finite correction to the energy of the configuration. We leave this problem open for later study. Such an analysis is needed for a complete evaluation of the leading quantum correction to the pair configuration energy.

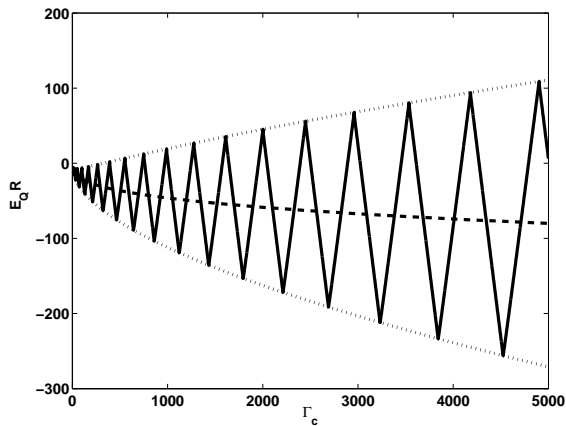


FIG. 3: The energy difference E_Q between the $m\bar{m}$ vacuum and the trivial vacuum is found to oscillate as a function of the number of modes (Γ_c) included in the energy sum for each sector. Also drawn in are the enveloping curves for the oscillations and the mean.

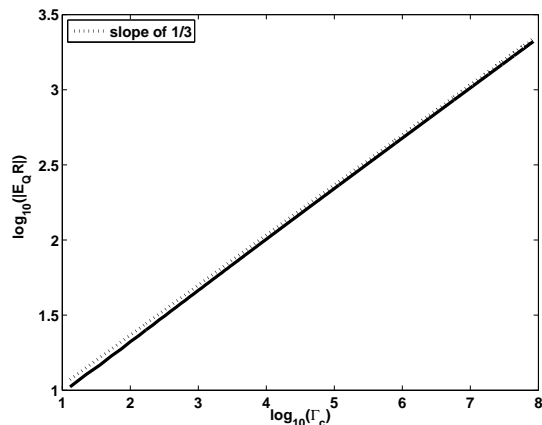


FIG. 4: Smoothing the oscillatory behavior shown in Fig. 3 reveals that on average $E_Q R$ varies as $-\Gamma_c^{1/3}$ as shown in this plot. The smoothing was done as explained in the text.

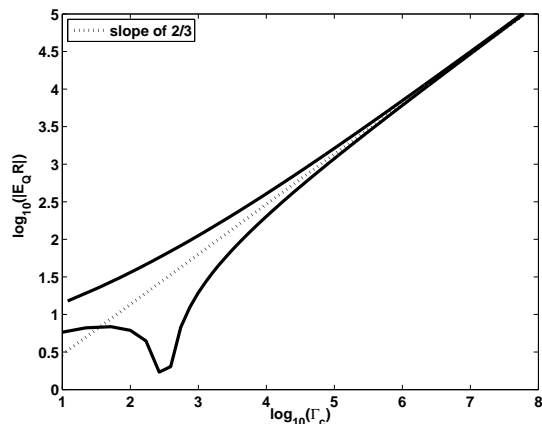


FIG. 5: The amplitude of the oscillations shown in Fig. 3 is shown by this logarithmic plot to vary as $\Gamma_c^{2/3}$.

VIII. CONCLUSIONS AND DISCUSSION

We have evaluated the complete set of vector spherical harmonics in the background of a monopole-antimonopole pair on the antipodes of an S^3 . We have also solved the radial problem and thus have a complete solution to the fluctuation problem. From our results, we can confirm the Brandt and Neri instability for $g \geq 1$ non-Abelian monopoles [14]. The unstable eigenmodes are determined explicitly and are found to be asymmetrical in the θ direction, suggesting an instability which might have an interpretation in terms of flux confinement.

We show that non-Abelian monopoles with minimum charge ($g = 1/2$) do not have a classical instability. We have constructed the complete set of fluctuation modes and their eigenvalues, and we use these results to find the energy of the zero point fluctuations over and above the energy of such fluctuations in the trivial vacuum. The resulting energy contribution E_Q is found to oscillate with an amplitude that grows as $\Gamma_c^{2/3}$ where Γ_c is the number of modes that we include in the sum over fluctuations. The mean value of E_Q however is proportional to $-\Gamma_c^{1/3}$. If the number of modes in the sum is limited by an ultra-violet momentum cutoff, P , then $\Gamma_c = PR$ and we find that $E_Q \propto -R^{-2/3}$. In other words, the mean contribution of the vacuum fluctuations increases with R but goes to zero in the $R \rightarrow \infty$ limit. For confinement we would expect $E_Q \sim R$ in the $R \rightarrow \infty$ limit and so we conclude that our results do not provide evidence for quantum confinement of non-Abelian magnetic monopoles, at least on S^3 .

Our analysis does indicate that quantum effects become important at large R because the zero point energy $\propto -R^{-2/3}$ goes to zero more slowly than the Coulomb energy $\propto -1/R$. At some critical value of R , quantum effects overtake classical effects. However, such a conclusion relies on the assumption that a full treatment of the renormalization of the monopole mass does not cancel out the $-R^{-2/3}$ dependence. In order to check this assumption, we would need to quantize regular non-Abelian monopoles *e.g.* $SU(3)$ monopoles as discussed in the introduction.

A puzzling feature of our analysis is the occurrence of oscillations in E_Q as a function of the mode cutoff Γ_c . We are convinced that these are present in our scheme of comparing energies in the $m\bar{m}$ and trivial sectors by taking an equal number, Γ_c , of lowest lying states in either sector. The underlying reason is that the degeneracy structure of the modes is different with and without monopoles. Mode degeneracies depend on the symmetries of the system and it is clear that the $m\bar{m}$ and trivial sectors have different symmetries under rotations. So it is not surprising that the degeneracies in the two cases are different. Once we accept the existence of different degeneracies, and we compare sums with the same number of modes, then the steps in the principal quantum number, n , in the sums over modes, arise at different values

of Γ_c for the cases with and without the monopoles. For some Γ_c the sums with the monopoles give a larger result and for other Γ_c the sum in the trivial vacuum is larger. While we understand the oscillations mathematically, we cannot rule out the possibility that if we were to track the flow of modes in transitioning from one sector to the other, say by bringing the monopoles closer and letting them annihilate, it may lead to a different subtraction scheme. We hope that future work will shed more light on these issues.

Finally we wish to remark that the quantum interaction of non-Abelian magnetic monopoles may be amenable to analysis by lattice methods and this would be an alternative approach to solving the problem.

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APPENDIX A: DIFFERENTIAL OPERATORS

In this appendix, we give the explicit form of Eqs. (11)-(16).

1. Neutral sector operators

Below are the equations of the uncharged sector. The (3) index of the vector components a_i is suppressed. The operator given in Eq. (A1) is used in the gauge condition, and the operator in equation (A2) is used in the equation of motion for the scalar component. Eqs. (A3)-(A5) are the components of the vector Laplacian, used in the

vector equations of motion.

$$\nabla^{(3)} \mathbf{a} = \partial_\chi a_\chi + 2 \frac{\cos \chi}{\sin \chi} a_\chi + \frac{1}{\sin^2 \chi} \left(\partial_\theta a_\theta + \frac{\cos \theta}{\sin \theta} a_\theta + \frac{1}{\sin^2 \theta} \partial_\phi a_\phi \right) \quad (\text{A1})$$

$$\Delta^{(3)} = \partial_\chi^2 + 2 \frac{\cos \chi}{\sin \chi} \partial_\chi + \frac{1}{\sin^2 \chi} \left(\partial_\theta^2 + \frac{\cos \theta}{\sin \theta} \partial_\theta + \frac{1}{\sin^2 \theta} \partial_\phi^2 \right) \quad (\text{A2})$$

$$(\Delta^{(3)} \mathbf{a})_\chi = \left(\Delta^{(3)} - \frac{2}{\sin^2 \chi} \right) a_\chi - 2 \frac{\cos \chi}{\sin^3 \chi} \left[\left(\partial_\theta + \frac{\cos \theta}{\sin \theta} \right) a_\theta + \frac{1}{\sin^2 \theta} \partial_\phi a_\phi \right] \quad (\text{A3})$$

$$(\Delta^{(3)} \mathbf{a})_\theta = \left(\Delta^{(3)} - \frac{1}{\sin^2 \chi \sin^2 \theta} \right) a_\theta + 2 \frac{\cos \chi}{\sin \chi} \left(\partial_\theta a_\chi - \partial_\chi a_\theta \right) - \frac{2}{\sin^2 \chi \sin^2 \theta} \frac{\cos \theta}{\sin \theta} \partial_\phi a_\phi \quad (\text{A4})$$

$$(\Delta^{(3)} \mathbf{a})_\phi = \Delta^{(3)} a_\phi + 2 \frac{\cos \chi}{\sin \chi} \left(\partial_\phi a_\chi - \partial_\chi a_\phi \right) - \frac{2}{\sin^2 \chi} \frac{\cos \theta}{\sin \theta} \left(\partial_\theta a_\phi - \partial_\phi a_\theta \right) \quad (\text{A5})$$

2. Charged sector operators

In the charged sector, derivatives are gauged, and Zeeman-like terms appear. We use the notation of $\mathcal{A}_\phi = (1 - \cos \theta)$, $g = eq/4\pi$, and $D_i \equiv \partial_\phi - ig\mathcal{A}_\phi$. Again, the

(+) index of the vector components a_i is suppressed.

$$\nabla^{(+)} \mathbf{a} = \partial_\chi a_\chi + 2 \frac{\cos \chi}{\sin \chi} a_\chi + \frac{1}{\sin^2 \chi} \left(\partial_\theta a_\theta + \frac{\cos \theta}{\sin \theta} a_\theta + \frac{1}{\sin^2 \theta} D_\phi a_\phi \right) \quad (\text{A6})$$

$$\Delta^{(+)} = \partial_\chi^2 + 2 \frac{\cos \chi}{\sin \chi} \partial_\chi + \frac{1}{\sin^2 \chi} \left(\partial_\theta^2 + \frac{\cos \theta}{\sin \theta} \partial_\theta + \frac{1}{\sin^2 \theta} D_\phi^2 \right) \quad (\text{A7})$$

$$(\Delta^{(+)} \mathbf{a})_\chi = \left(\Delta^{(+)} - \frac{2}{\sin^2 \chi} \right) a_\chi - 2 \frac{\cos \chi}{\sin^3 \chi} \left[\left(\partial_\theta + \frac{\cos \theta}{\sin \theta} \right) a_\theta + \frac{1}{\sin^2 \theta} D_\phi a_\phi \right] \quad (\text{A8})$$

$$\begin{aligned} (\Delta^{(+)} \mathbf{a})_\theta &= \left(\Delta^{(+)} - \frac{1}{\sin^2 \chi \sin^2 \theta} \right) a_\theta \\ &+ 2 \frac{\cos \chi}{\sin \chi} \left(\partial_\theta a_\chi - \partial_\chi a_\theta \right) \\ &- \frac{2}{\sin^2 \chi \sin^2 \theta} \left(\frac{\cos \theta}{\sin \theta} D_\phi - ig \partial_\theta \mathcal{A}_\phi \right) a_\phi \end{aligned} \quad (\text{A9})$$

$$\begin{aligned} (\Delta \mathbf{a}^{(+)})_\phi &= \Delta^{(+)} a_\phi + 2 \frac{\cos \chi}{\sin \chi} \left(D_\phi a_\chi - \partial_\chi a_\phi \right) \\ &- \frac{2}{\sin^2 \chi} \left[\frac{\cos \theta}{\sin \theta} \left(\partial_\theta a_\phi - D_\phi a_\theta \right) \right. \\ &\left. + ig [\partial_\theta \mathcal{A}_\phi] a_\theta \right] \end{aligned} \quad (\text{A10})$$

Note that in the case that $g = 0$ the equations of the charged sector coincide with those of the neutral sector.

APPENDIX B: DERIVATION OF EIGENMODES

1. Neutral scalar sector

We wish to find the eigenmodes of the scalar wave equation on S^3 . The obvious ansatz

$$a_t^{(3)} = \xi(\chi) Y_{JM}(\theta, \phi) e^{i\omega t} \quad (\text{B1})$$

leads to the eigenvalue equation

$$\left(\partial_\chi^2 + 2 \frac{\cos \chi}{\sin \chi} \partial_\chi + \left[\Lambda^2 - \frac{J(J+1)}{\sin^2 \chi} \right] \right) \xi = 0. \quad (\text{B2})$$

Making the substitutions $x = \cos \chi$ and $\xi = (1 - x^2)^{J/2} \eta$ leads to

$$\left(\partial_x^2 - (2J+3) \frac{x}{1-x^2} \partial_x + \frac{[\Lambda^2 - J(J+2)]}{1-x^2} \right) \eta = 0. \quad (\text{B3})$$

This is Gegenbauer's equation which has solutions regular at $x = \pm 1$ only for the quantized values $\Lambda^2 = n(n+2)$. These solutions are the Gegenbauer polynomials $G_{n-J}^{J+\frac{1}{2}}(x)$ where $n-J$ must be a non-negative integer. Thus we obtain the solutions $\xi_{n-J,J}(\chi) Y_{JM}(\theta, \phi) \exp(i\omega t)$ where $\xi_{n-J,J}$ is given by Eq. (18) in the text. The quantum numbers are restricted to $n = 0, 1, \dots$ and $J = 0, 1, \dots, n$ and $M = -J, \dots, J$.

As noted above the boundary conditions we impose are that the solution should be regular at $\chi = 0$ and $\chi = \pi$. The physical basis for these boundary conditions is that in the vicinity of the monopole, $\chi = 0$, the background field strength diverges as $(F^{\mu\nu} F_{\mu\nu})^{1/2} \propto \sin^{-2} \chi$, and there is a similar divergence in the vicinity of the antimonopole, $\chi = \pi$. In order for the perturbative expansion used in Eq. (8) to be valid, the divergence of the field perturbation $f_{\mu\nu}$ cannot be stronger than the background divergence. If the divergence is weaker than the background then

$$\lim_{\chi \rightarrow 0} \left(\frac{f^{\mu\nu} f_{\mu\nu}}{F^{\mu\nu} F_{\mu\nu}} \right) \rightarrow 0, \quad (\text{B4})$$

or if it is as strong as the background solution, then

$$\lim_{\chi \rightarrow 0} \left(\frac{f^{\mu\nu} f_{\mu\nu}}{F^{\mu\nu} F_{\mu\nu}} \right) \rightarrow \text{Const}. \quad (\text{B5})$$

In practice we find that the regular solutions to Gegenbauer's equation satisfy these boundary conditions; the singular ones do not. In fact the regular solutions all prove to be less singular than the background and so satisfy the more stringent boundary condition (B4). The boundary conditions we impose in the other sectors are similarly motivated.

For the trivial background which doesn't have $F_{\mu\nu}$, we demand regularity at $\chi = 0$ and $x = \pi$.

2. Charged scalar sector

In this sector we wish to solve the scalar wave equation with minimal coupling to the background vector potential of the monopole-antimonopole pair. The solution closely parallels the neutral case. The natural ansatz is to replace spherical harmonics with Wu-Yang monopole spherical harmonics:

$$a_t^{(+)} = \xi(\chi) W_{JM}(\theta, \phi) e^{i\omega t}. \quad (\text{B6})$$

This leads to the same equation for ξ as in the neutral case, Eq. (B2), but with the replacement $J(J+1) \rightarrow J(J+1) - g^2$. To bring out the parallel it is convenient to define α as in Eq. (21) of the text, so that $\alpha(\alpha+1) = J(J+1) - g^2$ and $\alpha \geq 0$. Next we substitute $x = \cos \chi$ and $\xi = (1 - x^2)^{\alpha/2} \eta$ to find that η obeys Gegenbauer's Eq. (B3) with the replacement $J \rightarrow \alpha$. Solutions that are regular at both $x = \pm 1$ can only be found for the

quantized values $\omega^2 = (n - J + \alpha)(n - J + \alpha + 2)$. These solutions are Gegenbauer polynomials $G_{n-J}^{\alpha+\frac{1}{2}}$ where $n - J$ is a non-negative integer.

In summary the solutions in the charged sector are $\xi_{n-J,\alpha}(\chi)W_{JM}(\theta, \phi)\exp(i\omega t)$ with frequencies $\omega^2 = (n - J + \alpha)(n - J + \alpha + 2)$. The quantum numbers are restricted to the range $n = g, g+1, g+2, \dots$ and $J = g, g+1, \dots, n$ and $M = -J, \dots, J$. The condition on n arises from the Gegenbauer quantization condition; the limits on J and M are the usual ones in the theory of monopole harmonics.

3. Neutral vector sector

We wish to determine the eigenmodes of the vector wave equation on S^3 . It is helpful to first solve the corresponding problem on R^3 . It is natural to seek solutions of the form $f(r)\mathbf{X}_{JM}^{(0)}(\theta, \phi)$, $f(r)\mathbf{X}_{JM}^{(+)}(\theta, \phi)$, and $f(r)\mathbf{X}_{JM}^{(-)}(\theta, \phi)$. The three families correspond to eigenmodes of different polarization. Here $\mathbf{X}_{JM}(\theta, \phi)$ are the vector spherical harmonics [22]. They are eigenfunctions of total angular momentum (orbital plus spin) with quantum numbers J and M . They are also eigenfunctions of orbital angular momentum squared with eigenvalues $J(J+1)$, $(J+1)(J+2)$ and $(J-1)J$ respectively. The quantum numbers span the range $J = 0, 1, 2, \dots$ and $M = -J, \dots, +J$. The case $J = 0$ is special in that there is only one vector spherical harmonic $\mathbf{X}_{00}^{(+)}$; the other two $\mathbf{X}_{00}^{(0)}$ and $\mathbf{X}_{00}^{(-)}$ vanish.

By use of the wave equation we determine that the radial functions $f(r)$ are spherical Bessel functions of order $J, J+1$ and $J-1$ respectively. Thus in polar co-ordinates the solutions designated $f(r)\mathbf{X}_{JM}^{(0)}(\theta, \phi)$ are given by

$$\begin{aligned} v_r &= 0 \\ v_\theta &= rj_J(\omega r)\frac{1}{\sin\theta}\partial_\phi Y_{JM} \\ v_\phi &= -rj_J(\omega r)\sin\theta\partial_\theta Y_{JM}; \end{aligned} \quad (B7)$$

the solutions $f(r)\mathbf{X}_{JM}^{(+)}$ by

$$\begin{aligned} v_r &= -(J+1)j_{J+1}(\omega r)Y_{JM} \\ v_\theta &= rj_{J+1}(\omega r)\partial_\theta Y_{JM} \\ v_\phi &= rj_{J+1}(\omega r)\partial_\phi Y_{JM}; \end{aligned} \quad (B8)$$

and the solutions designated $f(r)\mathbf{X}_{JM}^{(-)}$ by

$$\begin{aligned} v_r &= j_{J-1}(\omega r)JY_{JM} \\ v_\theta &= rj_{J-1}(\omega r)\partial_\theta Y_{JM} \\ v_\phi &= rj_{J-1}(\omega r)\partial_\phi Y_{JM}. \end{aligned} \quad (B9)$$

The frequency ω^2 is continuous and restricted to the range $\omega^2 \geq 0$.

For the subsequent generalization to S^3 it is necessary to introduce a different fundamental set of solutions. In

the new set, the first family of eigenmodes is obtained by taking the gradient of the modes of the scalar Laplacian on R^3 , namely $j_J(\omega r)Y_{JM}$. For the second family we adopt the transverse solutions $f(r)\mathbf{X}_{JM}^{(0)}$ given in Eq. (B7). For the third family we take the curl of the second group of transverse solutions. Explicitly, then the first family of solutions are

$$\begin{aligned} v_r &= \partial_r j_J(\omega r)Y_{JM} \\ v_\theta &= j_J(\omega r)\partial_\theta Y_{JM} \\ v_\phi &= j_J(\omega r)\partial_\phi Y_{JM}. \end{aligned} \quad (B10)$$

The second family is given by Eq. (B7). The third family of solutions, obtained by taking the curl of the second family, are

$$\begin{aligned} v_r &= \frac{1}{r}j_J(\omega r)J(J+1)Y_{JM} \\ v_\theta &= \partial_r[rj_J(\omega r)]\partial_\theta Y_{JM} \\ v_\phi &= \partial_r[rj_J(\omega r)]\partial_\phi Y_{JM}. \end{aligned} \quad (B11)$$

In the special case $J = 0$ the transverse solutions and their curls vanish; only the gradient solutions survive. This is consistent with the expectation that there is only one polarization in this exceptional angular momentum channel.

In summary the first independent set of solutions is given by Eqs. (B7), (B8) and (B9); the second set by Eqs. (B10), (B7) and (B11). By making use of recursion relations for Bessel functions we can show that the gradient solutions Eq. (B10) are the superposition $f(r)\mathbf{X}_{JM}^{(+)} + f(r)\mathbf{X}_{JM}^{(-)}$; the curl solutions Eq. (B11) are the superposition $-Jf(r)\mathbf{X}_{JM}^{(+)} + (J+1)f(r)\mathbf{X}_{JM}^{(-)}$. Thus the two alternative sets of eigenmodes are seen to be equivalent.

We are now ready to tackle the problem on S^3 . For the first set of eigenmodes we try the gradient of the scalar modes leading to the ansatz

$$\begin{aligned} a_\chi^{(3)} &= \partial_\chi \xi_{nJ} Y_{JM} \\ a_\theta^{(3)} &= \xi_{nJ} \partial_\theta Y_{JM} \\ a_\phi^{(3)} &= \xi_{nJ} \partial_\phi Y_{JM} \end{aligned} \quad (B12)$$

By applying the S^3 vector Laplacian to these modes we determine that their eigenvalues are $n(n+2)$.

For the second set of solutions we seek purely transverse solutions. By analogy to Eq. (B7) we make the ansatz

$$\begin{aligned} a_\chi^{(3)} &= 0 \\ a_\theta^{(3)} &= f(\chi)\frac{1}{\sin\theta}\partial_\phi Y_{JM} \\ a_\phi^{(3)} &= -f(\chi)\sin\theta\partial_\theta Y_{JM} \end{aligned} \quad (B13)$$

Application of the S^3 vector Laplacian shows that these functions are eigenmodes if we take $f(\chi) \rightarrow \sin\chi\xi_{nJ}(\chi)$.

These modes are found to have frequency $\omega^2 = n(n+2) + 1$.

The third set of solutions are obtained by taking the curl of the second transverse set. Again application of the S^3 vector Laplacian shows that the resulting modes are eigenfunctions with frequency $\omega^2 = n(n+2) + 1$.

By analogy to R^3 we see that for the second and third set of solutions $n = 1, 2, 3, \dots$ and $J = 1, 2, 3, \dots$ and $M = -J, \dots, J$. For the first set, $n = 1, 2, 3, \dots$ and $J = 0, 1, 2, \dots$ and $M = -J, \dots, J$.

This completes the derivation of the modes enumerated in section IV. Note it is possible to systematically rederive these eigenmodes for the vector Laplacian on S^3 using the theory of $so(4)$ representations just as in the scalar case. Although we will derive the relevant symmetry generators below we will not carry out this analysis here.

4. Charged vector sector

As noted in Sec. III, the eigenmode problem in the charged vector sector may be interpreted as the non-relativistic Schrödinger equation for a spin one particle confined to a 3-sphere and moving in the magnetic field of a monopole-antimonopole pair placed at opposite poles of the 3-sphere. The particle is minimally coupled to the magnetic field due to its charge as well as Zeeman coupled to the magnetic field due to its magnetic moment.

Again, as a prelude, let us solve the simpler problem of such a particle moving in R^3 in the field of a single monopole at the origin, a problem first investigated by Brandt and Neri [14]. In ref [14] the focus was on finding unstable modes by looking for imaginary eigenfrequencies. We extend that work by enumerating all stable modes and deriving explicit expressions for the eigenfunctions. Our primary interest in the R^3 problem is that it provides an important testing ground for the ansatzes that we will later deploy on S^3 .

In the absence of the Zeeman term it is natural to seek three families of solutions of the form $f(r)\mathbf{X}_{JM}^{(0)}(\theta, \phi)$, $f(r)\mathbf{X}_{JM}^{(+)}(\theta, \phi)$, and $f(r)\mathbf{X}_{JM}^{(-)}(\theta, \phi)$. Here $\mathbf{X}_{JM}(\theta, \phi)$ are the monopole vector spherical harmonics [16]. They are eigenfunctions of the total angular momentum (orbital plus spin) with quantum numbers J and M . They are also eigenfunctions of the total orbital angular momentum with eigenvalue $J(J+1)$, $(J+1)(J+2)$ and $(J-1)J$ respectively. For $g \geq 1$ the quantum numbers span the range $J = g-1, g, g+1, \dots$ and $M = -J, \dots, J$. However for $J = g-1$ there is only one vector spherical harmonic $\mathbf{X}^{(+)}$; for $J = g$ there are two polarizations $\mathbf{X}^{(+)}$ and $\mathbf{X}^{(0)}$. For all higher values, $J = g+1, g+2, \dots$, all three polarizations exist. Similarly for $g = 1/2$ the quantum numbers span the range $J = 1/2, 3/2, \dots$ and $M = -J, \dots, J$. However for $J = 1/2$ there are only two polarizations $\mathbf{X}^{(+)}$ and $\mathbf{X}^{(0)}$. For all higher values, $J = 3/2, 5/2, \dots$, all three polarizations exist.

The Zeeman term couples spin and orbital motion but

is invariant under total angular momentum. By the Wigner-Eckart theorem therefore it can only couple total angular momentum multiplets with the same quantum numbers. Thus we conclude that for a given J and M there will still be three families of solutions but they will be superpositions of the form

$$f(r) \left[a \mathbf{X}_{JM}^{(0)} + b \mathbf{X}_{JM}^{(+)} + c \mathbf{X}_{JM}^{(-)} \right]. \quad (B14)$$

The coefficients a, b, c are constants not only in that they are independent of (r, θ, ϕ) but also of M (in accordance with the Wigner-Eckart theorem). They are computed by solving for the eigenvectors of a 3×3 matrix that is determined by substituting the ansatz, Eq. (B14), into the eigenvalue equation for the charged vector sector. This matrix is given explicitly by Brandt and Neri; it can be efficiently derived making use of identities given in Ref. [16]. Substitution of the ansatz, Eq. (B14), into the eigenvalue equation also shows that $f(r)$ satisfies the radial equation for a spinless free particle in non-relativistic quantum mechanics,

$$\left(\partial_r^2 + \frac{2}{r} \partial_r - \frac{l(l+1)}{r^2} + \Lambda^2 \right) f = 0. \quad (B15)$$

with the modification that the angular momentum $l \rightarrow \alpha - 1$ or α or $\alpha + 1$ corresponding to the three families of solutions. For definiteness we shall call these families s, t and u respectively. Here α is given by Eq. (21) of the text.

Explicitly the solutions are given by

$$\begin{aligned} v_r &= [J(J+1) - g^2] j_{\alpha-1} W \\ v_\theta &= (\alpha+1) r j_{\alpha-1} \partial_\theta W \\ v_\phi &= (\alpha+1) r j_{\alpha-1} D_\phi W \end{aligned} \quad (B16)$$

for the s -polarization;

$$\begin{aligned} v_r &= 0 \\ v_\theta &= r j_\alpha \left([J(J+1) - g^2] \frac{i}{\sin \theta} D_\phi W + g \partial_\theta W \right) \\ v_\phi &= r j_\alpha (g D_\phi W - [J(J+1) - g^2] i \sin \theta \partial_\theta W) \end{aligned} \quad (B17)$$

for the t -polarization; and

$$\begin{aligned} v_r &= -[J(J+1) - g^2] j_{\alpha+1} W \\ v_\theta &= r j_{\alpha+1} \partial_\theta W \\ v_\phi &= r j_{\alpha+1} D_\phi W \end{aligned} \quad (B18)$$

for the u -polarization. Here $j_\alpha(\Lambda r)$ are spherical Bessel functions of order α . For brevity we have written the Wu-Yang harmonics $W_{JM}(\theta, \phi)$ as W in Eqs. (B16), (B17) and (B18). These modes form a continuum with frequency $\Lambda^2 > 0$. These three families of solutions are analogous respectively to Eqs. (B9), (B7) and (B8) to which they reduce for $g = 0$. The t -modes derive their name from being transverse.

There are two complications we have overlooked in Eqs. (B17), (B16) and (B18). First we have ignored the special values of J for which there are fewer than three polarizations. These special cases can be analyzed similarly. For example for case that $J = g$ we must drop the c term in the ansatz since there is no vector spherical harmonic $\mathbf{X}^{(-)}$ in that angular momentum channel. The upshot is that there are only two families of solutions in this angular momentum channel. These turn out to be the s and u solutions given above. It can be shown that the t solution is identically zero for $J = g$. For the case $J = g - 1$ we must drop the b and c in the ansatz Eq. (B14) and there is only one family of solutions. These solutions have the form

$$\begin{aligned} v_r &= 0 \\ v_\theta &= r f \exp(i[M + g]\phi) (\sin \theta)^{g+M-1} (1 + \cos \theta)^{-M} \\ v_\phi &= i r f \exp(i[M + g]\phi) (\sin \theta)^{g+M} (1 + \cos \theta)^{-M}. \end{aligned} \quad (\text{B19})$$

Here $M = -(g-1), \dots, (g-1)$ and $f(r)$ obeys Eq. (B15) with $l(l+1) = -g$. The angular factors in Eq. (B19) are taken from the angular dependence of the vector spherical harmonics $\mathbf{X}_{g-1,M}^{(+)}(\theta, \phi)$.

The second complication is that in non-relativistic quantum mechanics $l(l+1)$ is a non-negative integer and the term involving l in Eq. (B15) may be interpreted as a centrifugal barrier. Mathematically this means there are no bound state solutions; the continuum states are spherical Bessel functions. Here since we replace $l \rightarrow \alpha - 1$ or α or $\alpha + 1$ it is a concern that $l(l+1)$ may become negative depending on the particular value of α . For the negative case Eq. (B15) would have the interpretation of a particle in an attractive inverse cube central force potential. According to standard lore [23] in this case for a weakly attractive potential, one for which $0 > l(l+1) > -1/4$, there are no bound states and the unbound continuum states are still simply spherical Bessel functions. For $l(l+1) < -1/4$ there is a continuum of bound states with $\Lambda^2 < 0$ as well as a continuum of unbound states with $\Lambda^2 > 0$.

The continuum of bound states is an artifact of treating the monopole as a point. If we worked with a model in which the monopole had structure the singularity in the attractive inverse square potential would be softened at the origin. Presumably this would lead to a discrete bound state spectrum which is generally expected for non-singular potentials.

For $J = g$, we see that $l(l+1)$ can indeed be negative for the s polarized states for $g = 1/2, 1$ and $3/2$. However in all these cases it is not so negative as to form bound states (recall that for an s polarized state $l \rightarrow \alpha - 1$ where α is given by Eq. (21). It is now easy to verify that $-1/4 < l(l+1) < 0$ for $0 < g < 2$; otherwise $l(l+1)$ is positive). This leads to the important conclusion that $g = 1/2$ monopoles are stable (recall bound states correspond to instability). For $J = g - 1$ however $l(l+1) \rightarrow -g$ allowing the formation of bound states. Thus

monopoles with $g \geq 1$ exhibit instability in the $J = g - 1$ channel.

In summary Eqs. (B16), (B17) and (B18) are the eigenmodes needed to analyze the stability of a monopole in \mathbf{R}^3 . For $g = 1/2$ they are stable and represent a complete enumeration of modes. For $g \geq 1$ there is an additional branch of unstable modes with $J = g - 1$.

The chief virtue of the derivation above is that it is systematic and can be counted upon to yield a complete set of solutions. As in the neutral vector sector we now rederive the eigenmodes in a second more intuitive manner. The merit of this second derivation is that it produces slightly simpler expressions and generalizes readily to \mathbf{S}^3 . This second set of eigenmodes is also organized into three families. The first family is obtained by starting with the gradient solutions of the neutral vector sector, Eq. (B10) and making the minimal coupling substitution $\partial_\phi \rightarrow D_\phi$, replacing the spherical harmonics with Wu-Yang harmonics and substituting $J(J+1) \rightarrow J(J+1) - g^2$. Finally we need to replace the Bessel functions $j_J(\omega r)$ with $j_\alpha(\omega r)$. The third family is obtained by making the same replacements to the curl solutions of the neutral vector sector, Eq. (B11). Curiously the same substitutions made to the transverse solution of the neutral sector do not lead to a solution in the charged sector. For the second family we therefore retain the t modes of Eq. (B17).

We are now ready to tackle the problem on \mathbf{S}^3 . For the first family of eigenmodes we start with the gradient solution of the neutral vector sector Eq. (B12) and replace spherical harmonics with Wu-Yang harmonics and make the minimal coupling substitution $\partial_\phi \rightarrow D_\phi$. This is found to be a solution in the charged sector provided we also modify the radial function $\xi_{n-J,J}(\chi) \rightarrow \xi_{n-J,\alpha}(\chi)$. The eigenvalues are found to be $(n - J + \alpha)(n - J + \alpha + 2)$. For the third family we work with the curl solution of the neutral vector sector Eq. (26) and in addition to the same substitutions we replace $J(J+1) \rightarrow J(J+1) - g^2$. This procedure too yields a solution with eigenvalue $\Lambda^2 = (n + 1 - J + \alpha)^2$. Obtaining the second family of solutions is more challenging. We give an argument below that there have to be transverse eigenmodes. Bolstered by this argument we make the ansatz

$$\begin{aligned} a_\chi^{(+)} &= 0 \\ a_\theta^{(+)} &= [(J(J+1) - g^2) \frac{i}{\sin \theta} (D_\phi W) f + g \partial_\theta W f \\ a_\phi^{(+)} &= g D_\phi W f - i [J(J+1) - g^2] \sin \theta \partial_\theta W f \end{aligned} \quad (\text{B20})$$

f is a function of χ , W is the Wu-Yang harmonic $W_{JM}(\theta, \phi)$. This ansatz is motivated by the form of transverse modes in \mathbf{R}^3 and is found to yield solutions provided $f = \sin \chi \xi_{n-J,\alpha}(\chi)$. The eigenvalues are $(n + 1 - J + \alpha)^2$.

As in \mathbf{R}^3 it is possible to show that the second transverse family does not exist for $J = g$. There is also a single branch of unstable solutions for $J = g - 1$ for

$g \geq 1$. These solutions have the form

$$\begin{aligned} a_\chi &= 0 \\ a_\theta &= \sin \chi \xi \exp[i(M+g)\phi] (\sin \theta)^{g+M-1} (1+\cos \theta)^{-M} \\ a_\phi &= i \sin \chi \xi \exp[i(M+g)\phi] (\sin \theta)^{g+M} (1+\cos \theta)^{-M} \end{aligned}$$

where the radial function $\xi(\chi)$ satisfies Eq. (B2) with $J(J+1) \rightarrow -g$. A simple analysis reveals that both solutions to this equation diverge as $\xi \propto 1/\sqrt{\chi}$ as $\chi \rightarrow 0$ and similarly as $\chi \rightarrow \pi$. Since both solutions behave acceptably at the boundary points there is no quantization or restriction on the frequencies Λ^2 . Hence we conclude that in the $J = g - 1$ channel there is a continuum of bound states with $\Lambda^2 < 0$ and a continuum of unbound states with $\Lambda^2 \geq 0$.

It may seem counterintuitive that the spectrum of modes is continuous on a finite space. Similar to the R^3 case, this result is an artifact of treating the monopoles as points. If we had worked in a model in which the monopole cores had structure the singularity of the potential in the radial equation would be softened at the north and south pole of S^3 . This would lead to a discrete spectrum, which is generally expected for non-singular potentials. However we would still expect both bound and unbound modes.

This concludes the derivation of the charged sector solutions and the range of allowed quantum numbers that are given in section IV.

The main difficulty with solving Eqs. (A8), (A9) and (A10) is that they are a system of coupled partial differential equations. A puzzling aspect of our solution above is that we were able to obtain two solutions by making simple substitutions in the neutral vector solutions but the transverse solution did not yield to this strategy. For this reason we would like to present another line of argument that demonstrates there must be a transverse solution. This approach can also be applied to the neutral sector but here we concentrate on the more vexing charged case.

We begin with the gauge condition (16), re-written as

$$\partial_\theta a_\theta + \frac{\cos \theta}{\sin \theta} a_\theta + \frac{1}{\sin^2 \theta} D_\phi a_\phi = \sin^2 \chi \left(R_0 \partial_t a_t - \partial_\chi a_\chi - 2 \frac{\cos \chi}{\sin \chi} a_\chi \right), \quad (\text{B21})$$

where we have suppressed the (+) index. By substituting this in the χ component of the vector Laplacian, Eq. (A8), we eliminate the dependence on a_θ and a_ϕ ,

$$\begin{aligned} &\left(\partial_\chi^2 + 4 \frac{\cos \chi}{\sin \chi} \partial_\chi + (R \partial_t)^2 + \frac{2}{\sin^2 \chi} - 4 \right) a_\chi + \\ &\frac{1}{\sin^2 \chi} \left(\partial_\theta^2 + \frac{\cos \theta}{\sin \theta} \partial_\theta + \frac{1}{\sin^2 \theta} D_\theta^2 \right) a_\chi = R_0 \partial_t a_t. \end{aligned} \quad (\text{B22})$$

Following the physical analogy to the Schroedinger equation, we now look for 3 solutions to the above equation.

Repeatedly, our aim will be to somehow force separation of the χ and the (θ, ϕ) dependence. We will assume a trivial time dependence of the form $\exp[i\omega t]$. Given that time dependence and for a general ω , the above equation allows one of three options - a) both a_t and a_χ are non-zero, b) a_t is zero but a_χ isn't, or c) a_t and a_χ are zero.

A reasonable first ansatz is $a_t = C a_\chi$, C being a constant. We posit a separation of variables $a_\chi = P(\theta, \phi) f(\chi)$ in Eq. (B22), solve this equation for a_χ , then use the resulting solution in Eqs. (A9) and (A10) to determine a_θ and a_ϕ yielding the first solution Eq. (27).

Next we attempt a solution with $a_t = 0$. Eq. (B22) then gives a solution for a_χ . Armed with this solution we return to Eqs. (A9) and (A10) and posit that a_θ and a_ϕ have the same χ dependence. To be specific we try

$$\begin{aligned} a_\chi &= k P(\theta, \phi) \frac{f(\chi)}{\sin^2 \chi} e^{i\omega t} \\ a_\theta &= \frac{G(\theta, \phi)}{\sin \theta} g(\chi) e^{i\omega t} \\ a_\phi &= i \sin \theta H(\theta, \phi) g(\chi) e^{i\omega t} \end{aligned} \quad (\text{B23})$$

where k is a constant. Plugging Eq. (B23) into the gauge condition gives the relations that $g = \partial_\chi f$ and $k \sin \theta P + \partial_\theta G + i D_\phi H = 0$. Using these relations in Eqs. (A9) and (A10) gives the third solution Eq. (29).

Finally let us attempt a transverse ansatz

$$\begin{aligned} a_\chi &= 0 \\ a_\theta &= \frac{G(\theta, \phi)}{\sin \theta} g(\chi) e^{i\omega t} \\ a_\phi &= i \sin \theta H(\theta, \phi) g(\chi) e^{i\omega t}. \end{aligned} \quad (\text{B24})$$

Plugging this into the gauge condition, Eq. (B21) gives the relation that $\partial_\theta G = i D_\phi H$. We assume that $D_\phi H = i(\mathcal{M} + g\mathcal{A})H$. Substitution of these results into Eqs. (A9) and (A10) yields

$$\left[\partial_x^2 + \Lambda^2 \right] f(\chi) = \frac{N}{\sin^2 \chi} f(\chi) \quad (\text{B25})$$

$$\begin{aligned} &\left[\partial_\theta^2 + \left(\frac{\cos \theta}{\sin \theta} - 2g \frac{\partial_\theta \mathcal{A}}{\mathcal{M} + g\mathcal{A}} \right) \partial_\theta \right. \\ &\left. + N - \frac{(\mathcal{M} + g\mathcal{A})^2}{\sin^2 \theta} \right] G = 0 \end{aligned} \quad (\text{B26})$$

where we used $N = J(J+1) - g^2$ for brevity. While Eq. (B25) is easily transformed into the Gegenbauer equation, the angular Eq. (B26) is less readily solvable. Nonetheless because the variables separate, these two equations demonstrate the consistency of the transverse ansatz Eq. (B24). Eq. (B26) is essentially the Wu-Yang monopole harmonic equation but with an extra term $2g(\partial_\theta \mathcal{A})/(\mathcal{M} + g\mathcal{A})\partial_\theta$ that makes the equation difficult to solve. However we can *a posteriori* verify that the solution Eq. (28) is not only consistent with the transverse ansatz Eq. (B24) but also an explicit solution to Eqs. (B25) and (B26).

5. Zero-Modes

Here we use the group theoretic technique discussed in Appendix C to find the translational zero mode solutions.

First let us develop expressions for rotation generators acting on vector fields in \mathbb{R}^4 . If we initially represent vectors in terms of their cartesian components (v_1, v_2, v_3, v_4) then the generator of rotations in the x_3 - x_4 plane is given by

$$N_3 = i \cos \theta \partial_\chi - i \sin \theta \frac{\cos \chi}{\sin \chi} \partial_\theta + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix} \quad (\text{B27})$$

This result follows from Eqs. (C1) and (C6). We would rather work with polar components which are related to cartesian components via

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = U \begin{bmatrix} v_r \\ v_\chi \\ v_\theta \\ v_\phi \end{bmatrix} \quad (\text{B28})$$

where the transformation matrix U is given by

$$U = \begin{bmatrix} s\chi s\theta c\phi & c\chi s\theta c\phi/r & c\theta c\phi/rs\chi & -s\phi/rs\chi s\theta \\ s\chi s\theta s\phi & c\chi s\theta s\phi/r & c\theta s\phi/rs\chi & c\phi/rs\chi s\theta \\ s\chi c\theta & c\chi c\theta/r & -s\theta/rs\chi & 0 \\ c\chi & -s\chi/r & 0 & 0 \end{bmatrix} \quad (\text{B29})$$

Here for brevity we have written $\sin \theta = s\theta$, $\cos \chi = c\chi$ etc. In terms of polar components N_3 is given by $U^{-1}N_3(\text{cartesian})U$. A simple calculation reveals the polar form

$$N_3 = i \cos \theta \partial_\chi - i \sin \theta \frac{\cos \chi}{\sin \chi} \partial_\theta + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i \sin \theta / \sin^2 \chi & 0 \\ 0 & -i \sin \theta & -i \cos \theta \cos \chi & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (\text{B30})$$

We are interested in vectors that live on the unit sphere in \mathbb{R}^4 . Such vectors have no radial component and we may therefore discard the first row and column of the matrix in Eq. (B30). The 3×3 matrix differential operators corresponding to rotations in the other planes may be constructed similarly.

Now let us construct the zero modes. The background monopole field has polar components $A_\chi = A_\theta = 0$ and $A_\phi = g(1 - \cos \theta)$. The zero modes are obtained by considering the change in this field under infinitesimal rotation in the x_1 - x_4 , x_2 - x_4 and x_3 - x_4 planes, or, in other words, by application of N_1, N_2 and N_3 to the

background field configuration. Application of N_3 , for example, yields $(N_3 A)_\chi = (N_3 A)_\theta = 0$ and $(N_3 A)_\phi = -ig(\cos \chi / \sin \chi) \sin^2 \theta$ which is the first zero mode listed in section IV. The other two are obtained by application of N_1 and N_2 .

APPENDIX C: GROUP THEORETIC DERIVATION OF MODES IN NEUTRAL SCALAR SECTOR

It is instructive to rederive this result using the rotational symmetry of the problem. Readers who are not interested in a group theoretic rederivation may skip this section but should still examine Eqs. (C1) and (C6) and read the three sentences preceding Eq. (C1) and the entire paragraph containing Eq. (C6) as these results will be used to construct the zero mode solutions.

Picture the sphere S^3 embedded in a four-dimensional space. It is easy to write down the 4×4 matrices corresponding to rotation generators in the $x_1 - x_2$, $x_2 - x_3$, $x_3 - x_1$ and $x_1 - x_4$, $x_2 - x_4$, $x_3 - x_4$ planes. We denote these generators J_1, J_2, J_3 and N_1, N_2, N_3 respectively. For example

$$J_3 = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad N_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix} \quad (\text{C1})$$

J_3 and N_3 commute and together constitute a Cartan subalgebra. By computing the commutators of all six matrices we can obtain the structure constants of the $\text{so}(4)$ algebra. It is convenient to define

$$\begin{aligned} K_+ &= \frac{1}{2}(J_+ + N_+); \quad M_+ = \frac{1}{2}(J_+ - N_+) \\ K_- &= \frac{1}{2}(J_- + N_-); \quad M_- = \frac{1}{2}(J_- - N_-) \\ K_3 &= \frac{1}{2}(J_3 + N_3); \quad M_3 = \frac{1}{2}(J_3 - N_3) \end{aligned} \quad (\text{C2})$$

where, as usual, $J_+ = J_1 + iJ_2$ and $J_- = J_1 - iJ_2$ and the same for N . By using the explicit matrices it is easy to verify that \mathbf{K} and \mathbf{M} separately obey the angular momentum algebra and commute with each other. In other words $\text{so}(4) = \text{su}(2) \oplus \text{su}(2)$.

Working with the \mathbf{K}, \mathbf{M} generators, Eq. (C2), we see from the theory of angular momentum that representations of the $\text{so}(4)$ algebra can be labeled by l_1 and l_2 which may take any value from $0, \frac{1}{2}, 1, \frac{3}{2}, \dots$. The representation (l_1, l_2) has dimensionality $(2l_1+1)(2l_2+1)$. The basis vectors in this representation are denoted $|m_1, m_2\rangle$ where $m_1 = -l_1, \dots, l_1$ and $m_2 = -l_2, \dots, l_2$. The effect of the basic algebra elements in this representation

is given by

$$\begin{aligned}
K_3|m_1, m_2\rangle &= m_1|m_1, m_2\rangle \\
K_+|m_1, m_2\rangle &= [(l_1 + m_1 + 1)(l_1 - m_1)]^{\frac{1}{2}}|m_1 + 1, m_2\rangle \\
K_-|m_1, m_2\rangle &= [(l_1 + m_1)(l_1 - m_1 + 1)]^{\frac{1}{2}}|m_1 - 1, m_2\rangle \\
M_3|m_1, m_2\rangle &= m_2|m_1, m_2\rangle \\
M_+|m_1, m_2\rangle &= [(l_2 + m_2 + 1)(l_2 - m_2)]^{\frac{1}{2}}|m_1, m_2 + 1\rangle \\
M_-|m_1, m_2\rangle &= [(l_2 + m_2)(l_2 - m_2 + 1)]^{\frac{1}{2}}|m_1, m_2 - 1\rangle
\end{aligned} \tag{C3}$$

Instead of the $|m_1, m_2\rangle$ states it is sometimes convenient to work with the $|JM\rangle$ basis defined by

$$|JM\rangle = \sum_{m_1=-l_1}^{l_1} \sum_{m_2=-l_2}^{l_2} C_{m_1 m_2}^{JM} (l_1, l_2) |m_1, m_2\rangle \tag{C4}$$

where $J = |l_1 - l_2|, \dots, l_1 + l_2$ and $M = -J, \dots, J$. $C_{m_1 m_2}^{JM} (l_1, l_2)$ are the Clebsch-Gordan coefficients. There are $(2l_1 + 1)(2l_2 + 1)$ of these states, as there should be, since the $|JM\rangle$ states are simply an alternative basis for the (l_1, l_2) representation of the $\text{so}(4)$ algebra. In this basis the matrices for $\mathbf{J} = \mathbf{K} + \mathbf{M}$ are simple but not for \mathbf{N} or \mathbf{K} or \mathbf{M} .

The natural Casimir invariant for the $\text{so}(4)$ algebra is

$$\mathcal{C} = \mathbf{J}^2 + \mathbf{N}^2 = 2\mathbf{K}^2 + 2\mathbf{M}^2 \tag{C5}$$

All the states in an (l_1, l_2) representation are eigenstates of \mathcal{C} with eigenvalue $2l_1(l_1 + 1) + 2l_2(l_2 + 1)$.

Square representations where $l_1 = l_2 = l$ are of particular interest. It is helpful to define $n = 2l$. Thus a square representations is labeled by a single integer, n , its order. A square representation has dimensionality $(2l + 1)^2 = (n + 1)^2$. For square representations we see that in the $|JM\rangle$ basis the quantum numbers span the range $J = 0, \dots, n$ and $M = -J, \dots, +J$. Moreover the Casimir invariant is given by $n(n + 2)$.

Now let us consider wavefunctions $\psi(\theta, \phi, \chi)$ on S^3 . The Hilbert space of these wavefunctions constitutes a reducible infinite dimensional representation of the $\text{so}(4)$ algebra. By considering the rotation of a wavefunction we can deduce the generators of rotations in the six fundamental planes. For example

$$J_3 = -i\partial_\phi, \quad N_3 = i\cos\theta\partial_\chi - i\sin\theta\frac{\cos\chi}{\sin\chi}\partial_\theta. \tag{C6}$$

These differential operators obey the $\text{so}(4)$ algebra defined by their 4×4 matrix counterparts. In deriving the differential operators it is helpful to recall that the relation between the polar co-ordinates of a point on S^3 and its four-dimensional cartesian co-ordinates is

$$\begin{aligned}
x_1 &= \sin\chi \sin\theta \cos\phi \\
x_2 &= \sin\chi \sin\theta \sin\phi \\
x_3 &= \sin\chi \cos\theta \\
x_4 &= \cos\chi
\end{aligned} \tag{C7}$$

and that $J_3 = -ix_1\partial_2 + ix_2\partial_1$, $N_3 = -ix_3\partial_4 + ix_4\partial_3$.

A simple calculation reveals that the Casimir differential operator

$$\begin{aligned}
\mathcal{C} &= -\partial_\chi^2 - \frac{1}{\sin^2\chi}\partial_\theta^2 + \frac{1}{\sin^2\chi\sin^2\theta}\partial_\phi^2 \\
&+ 2\frac{\cos\chi}{\sin\chi}\partial_\chi + \frac{\cos\theta}{\sin\theta}\frac{1}{\sin^2\chi}\partial_\theta.
\end{aligned} \tag{C8}$$

coincides with the Laplacian on S^3 . Furthermore it is possible to verify that the eigenfunctions of the Laplacian $\xi_{nJ}(\chi)Y_{JM}(\theta, \phi)$ that we obtained by separation of variables have the following group theoretic interpretation: For a fixed n , they constitute the $|JM\rangle$ basis for a square representation of order n .

That only square representations are realized can be demonstrated by seeking functions that satisfy $K_+\psi = 0$ and $M_+\psi = 0$. These are first-order equations and the solutions are readily found to be

$$\psi_n = (\sin\chi)^n (\sin\theta)^n \exp(in\phi) \tag{C9}$$

where n is a non-negative integer. Since these are the highest weight states of the representation we can deduce the (l_1, l_2) values of the representation by application of K_3 and M_3 to ψ_n . By explicit calculation we find $K_3\psi_n = M_3\psi_n = (n/2)\psi_n$ revealing that the representations under consideration are indeed square with $l_1 = l_2 = n/2$.

APPENDIX D: BOUNDS ON E_Q

In order to derive the bound first consider the contribution of the charged sector in the absence of the monopole antimonopole pair. The number of modes with frequency less than or equal to $N + 1$ is clearly

$$\Gamma_N = \sum_{n=1}^N \sum_{J=1}^n \sum_{M=-J}^J 2 = \frac{1}{3}N(N+1)(2N+7). \tag{D1}$$

Let us denote the frequency of the Γ^{th} mode by $\Lambda(\Gamma)$. Then $\Lambda(\Gamma)$ is a staircase function that jumps at $\Gamma = \Gamma_N$ where $N = 1, 2, \dots$. Evidently $\Lambda(\Gamma) = N + 1$ for $\Gamma_{N-1} < \Gamma \leq \Gamma_N$. In terms of this staircase function the contribution to E_Q is given by $(1/2)\int_0^{\Gamma_c} d\Gamma \Lambda(\Gamma)$. We can derive upper and lower bounds by approximating the staircase $\Lambda(\Gamma)$ by smooth functions that intersect the top of each step and the bottom of each step respectively. Explicit calculation leads to the bounds

$$\begin{aligned}
RE_{\text{upper}} &= \frac{1}{4}\left(\frac{3}{2}\right)^{4/3} \Gamma_c^{4/3} + \frac{1}{4}\Gamma_c \\
RE_{\text{lower}} &= \frac{1}{4}\left(\frac{3}{2}\right)^{4/3} \Gamma_c^{4/3} - \frac{1}{4}\Gamma_c
\end{aligned} \tag{D2}$$

These expressions apply in the relevant limit $\Gamma_c \gg 1$.

The zero-point energy of the charged sector when the pair is present can be bounded similarly. We make the additional simplification that $\Lambda_{nJ} = n$ whilst deriving the lower bound and $\Lambda_{nJ} = n + 1$ whilst deriving the upper bound. The results are:

$$\begin{aligned} RE_{\text{upper}} &= \frac{1}{4} \left(\frac{3}{2} \right)^{4/3} \Gamma_c^{4/3} + \frac{3}{4} \Gamma_c \\ RE_{\text{lower}} &= \frac{1}{4} \left(\frac{3}{2} \right)^{4/3} \Gamma_c^{4/3} - \frac{1}{4} \Gamma_c \end{aligned} \quad (\text{D3})$$

Upper and lower bounds on E_Q may now be derived by taking the differences of the bounds in Eqs. (D2) and (D3). In the discussion so far we have concentrated upon a single charged sector. Since there are in fact two charged sectors, $+$ and $-$, we need to double the answer, leading to the result:

$$-\Gamma_c < RE_Q < 2\Gamma_c. \quad (\text{D4})$$

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